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Discussion on the Processing Methods of Curve Integrals and Surface Integrals

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ABSTRACT
This paper aims to explore common approaches to evaluating curve and surface integrals, including the definition-based method (direct computation), substitution techniques, and the use of symmetry properties, in order to simplify the computational workflow and enhance problem-solving skills in practical applications.

KEYWORDS
Curve integral; Surface integral; Substitution method; Arc length; Arc surface; Symmetry

1 Introduction

Curve integrals and surface integrals are core contents in advanced mathematics and mathematical analysis courses, widely applied in physics, engineering, economics, and other fields. However, due to their abstract concepts and complex calculations, many students often feel confused, especially when dealing with the diversity of integral paths or surfaces, symmetry, and the application of substitution methods. This paper systematically discusses the processing methods of curve integrals and surface integrals, including the definition method (conventional method), substitution method, and symmetry method, to help readers simplify the calculation process and improve their ability to solve practical problems.

2 Definition Method (Conventional Method)

2.1 Curve Integrals

Curve integrals are generally divided into two categories: line integrals of the first kind and line integrals of the second kind, which correspond respectively to integration with reference to arc length and integration with regard to coordinate variables. A key distinction lies in path dependence: first-type line integrals exhibit path independence, whereas the evaluation results of second-type line integrals are contingent upon the specific integration path selected. This implies that for the first kind, the upper limit must exceed the lower limit, while the second kind does not have this restriction—its limits correspond to the start and end points of the path.

In the context of mathematical analysis, the first-type line integral, from a physical perspective, denotes the total mass of a curve with a given density function. The arc-length differential element can be expressed in three distinct forms depending on the curve's representation:

For a curve defined by the explicit function $y = y(x)$, we first compute the derivative term $y = y(x)$, then derive:

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

Substituting this into the integral expression yields:

$$\int_L f(x,y)ds = \int_a^b f[x,y(x)] \sqrt{1 + (y'(x))^2}\, dx$$

When the curve is given in implicit form $x = x(y)$, we apply implicit differentiation to obtain:

$$ds = \sqrt{1 + (\frac{dx}{dy})^2}\, dy$$

This leads to the integral formulation:

$$\int_L f(x,y)ds = \int_c^d f[x(y),y] \sqrt{1 + (x'(y))^2}\, dy$$

For parametric curves represented by $x = x(t), y = y(t)$ where $x = x(t), y = y(t)$ are differentiable functions, the arc-length element becomes:

$$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$$

resulting in the parametric integral form:

$$\int_L f(x,y)ds = \int_m^n f[x(t),y(t)] \sqrt{(x'(t))^2 + y'(t)^2}\, dt$$

These expressions provide a straightforward means for directly computing curve integrals.

2.2 Surface Integrals

Similar to line integrals, surface integrals include surface integrals over area ($\iint_{\Sigma} f(x,y,z)dS$) and surface integrals over coordinates ($\iint_{\Sigma} X(x,y,z)dydz + Y(x,y,z)dzdx + Z(x,y,z)dxdy$). To properly handle surface integrals over area, the following two points need to be addressed:

(1) The surface integral over area $\iint_{\Sigma} f(x,y,z)dS$is physically interpreted as the mass of a smooth surface Σ with surface density$f(x,y,z)$. Therefore, when converting a surface integral over area to a surface integral over coordinates, it is necessary to ensure that the projected region of surface Σ is in the positive direction. (2) Use the implicit equation of surface $\Sigma F(x,y,z)=0$to handle the projected region problem. When converting the surface integral over area $\iint_{\Sigma} f(x,y,z)dS$to a surface integral over coordinates, there are three types of projected regions to choose from for surface Σ. The following takes the projected region D_{xy} as an example for processing. Let the surface Σ be:$F(x,y,z)=0$

Note the normal vector of $\Sigma \overrightarrow{n} = (F_x, F_y, F_z)$. Normalizing the normal vector $\overrightarrow{n}$, we have $|\cos\gamma| = \frac{|F_z|}{\sqrt{F_x^2 + F_y^2 + F_z^2}}$. Therefore, the surface integral over area can be converted to a surface integral over coordinates. When dealing with the line integral over coordinates $\iint_{\Sigma} X(x,y,z)dydz + Y(x,y,z)dzdx + Z(x,y,z)dxdy$, the key is to make good use of the two important roles of the oriented surface Σ.(1) Significance of the normal vector for oriented surface Σ.The surface integral over coordinates is interpreted as the flux of fluid with density 1 and velocity field $\vec{s}(x,y,z) = (X(x,y,z), Y(x,y,z), Z(x,y,z))$flowing through the specified side of Σ per unit time. The calculation of the surface integral over coordinates is related to the orientation of the oriented surface Σ, and the orientation of surface Σ is determined by its normal vector $\overrightarrow{n}$. Therefore, the surface integral over coordinates is closely related to the angles between the normal vector $\overrightarrow{n}$of surface Σ and the positive directions of the three coordinate axes. (2) Use the equation $z = z(x,y)$determined by surface Σ to solve the problem of excessive integral variables. For the surface integral over coordinates $\iint_{\Sigma} R(x,y,z)dxdy$ Despite the presence of merely two integral symbols in its notation, the surface integral necessitates the evaluation of three variables x, y, and z. The solution is that since the point(x,y,z)lies on the surface Σ, substitute the equation$z = z(x,y)$determined by surface Σ into the integral $\iint_{\Sigma} R(x,y,z)dxdy$.Note that the sign corresponding to the angle between the normal vector $\overrightarrow{n}$of surface Σ and the positive z-axis determines the appropriate sign, which allows converting the surface integral $\iint_{\Sigma} R(x,y,z)dxdy$into a double integral $\pm\iint_{\Sigma} R(x,y,z(x,y))dxdy$on the *xoy* plane for solution.

3 Substitution Technique

3.1 Substitution Approach for Curve Integrals

3.1.1 Substitution Approach for Line Integrals Along Plane Curves

Theorem 1 Consider a plane curve L defined by $\varphi(x,y)=0$ Suppose there exists a transformation $\begin{cases} x = x(m,n) \\ y = y(m,n) \end{cases}$ that bijection

maps the plane curve L'from the plane uOv to L in the plane uOv Herein, the function $\Phi(x, y)$ possesses first-order continuously differentiable partial derivatives over the domain D , while the functions x(u,v) and y(u,v) exhibit first-order continuously differentiable partial derivatives on the parameter domain Ω, with $(\varphi_1'^2 + \varphi_2'^2)\big|_L \neq 0$, $\left.\frac{\partial(x,y)}{\partial(m,n)}\right|_{L'} \neq 0$, and$f(x,y)$ is continuous onLUnder these conditions, the equation $\int_L f(x,y)ds = \int_{L'} f(x(m,n),y(m,n)) \frac{\sqrt{(\varphi_1'^2 + \varphi_2'^2)}}{\left\| J^{-1} \begin{pmatrix} \varphi_2' \\ -\varphi_1' \end{pmatrix} \right\|} ds_{ab}$ holds, whereds = $\sqrt{dx^2 + dy^2}$, $ds_{uv} = \sqrt{(dm)^2 + (dn)^2}$, $J = \begin{pmatrix} \frac{\partial x}{\partial m} & \frac{\partial x}{\partial n} \\ \frac{\partial y}{\partial m} & \frac{\partial y}{\partial n} \end{pmatrix}$

Corollary 1 When Jforms an orthogonal matrix, the relation $\int_L f(x,y)ds = \int_{L'} f(x(m,n),y(m,n))ds_{mn}$ holds

3.1.2 Substitution Approach for Line Integrals Along Spatial Curves

Theorem 2 Consider a spatial curve $\begin{cases}\varphi(e,f,g)=0\\ \psi(e,f,g)=0\end{cases}$ defined by its equation. Apply the transformation $\begin{cases}x=x(e,f,g)\\ y=y(e,f,g)\\ z=z(e,f,g)\end{cases}$ to bijection map the curve Γ'from O – uvw to Γ within O – xyz. In this context, the functions φ(x,y,z) and ψ(u,v,w) possess first - order continuously differentiable partial derivatives over the domain Ω, while the function χ(s, t) exhibits first - order continuously differentiable partial derivatives on the parameter domain D. Define $G(x,y,z)=\text{grad}\varphi*\text{grad}\psi$. Given $G(x,y,z)\big|_r \neq 0$, $\left.\frac{\partial(x,y,z)}{\partial(e,f,g)}\right|_{\Gamma'} \neq 0$, and f(x,y,z)being continuous on Γ the equation $\int_L f(x,y,z)ds = \int_\Gamma f(x(e,f,g),y(e,f,g),z(e,f,g))\frac{\|G(x,y,z)\|}{\|J^{-1}G(x,y,z)^T\|}ds_{efg}$ holds with $ds=\sqrt{(dx)^2+(dy)^2+(dz)^2}$, $ds_{efg}=\sqrt{(de)^2+(df)^2+(dg)^2}$,$J=\begin{Bmatrix}\frac{\partial x}{\partial e} & \frac{\partial x}{\partial f} & \frac{\partial x}{\partial g}\\ \frac{\partial y}{\partial e} & \frac{\partial y}{\partial f} & \frac{\partial y}{\partial g}\\ \frac{\partial z}{\partial e} & \frac{\partial z}{\partial f} & \frac{\partial z}{\partial g}\end{Bmatrix}$

Corollary 2: When J forms an orthogonal matrix, the relation $\int_L f(x,y,z)ds = \int_{\Gamma'} f(x(e,f,g),y(e,f,g),z(e,f,g))ds_{efg}$ holds

3.1.3 Variable Transformation Approach for Coordinate Line Integrals on Plane Curves

Theorem 3: Consider L as a smooth plane curve . Through the transformation $\begin{cases}x=x(a,b)\\ y=y(a,b)'\end{cases}$ the directed curve L'along with points $A^{'}$ and $B^{'}$ on the plane uOv are mapped to L and points Aand B on the plane xOy where L' starts at $A^{'}$ and ends at $B^{'}$. Assuming x(a,b) and y(a,b) possess first-order continuous partial derivatives $\left.\frac{\partial(x,y)}{\partial(a,b)}\right|_{L'} \neq 0$ on L', while P = P(x,y) and Q = Q(x,y)are continuous on L, the equation $\int_L P(x,y)dx+Q(x,y)dy = \int_{L'}(P\frac{\partial x}{\partial a}+Q\frac{\partial x}{\partial a})du+(P\frac{\partial x}{\partial b}+Q\frac{\partial y}{\partial b})dv$ holds

3.1.4 Coordinate Transformation Technique for Line Integrals Along Spatial Curves

Theorem 4: Consider Γ as a smooth oriented curve in space. Applying the transformation $\begin{cases}x=x(e,f,g)\\ y=y(e,f,g)\\ z=z(e,f,g)\end{cases}$, the oriented curve Γ′ and points P′, Q′ within O – uvw are mapped to Γ and points P, Q in O – xyz. Here, Γ′ starts at P′ and ends at Q′. Assuming x(e,f,g),y(e,f,g),z(e,f,g) possess first-order continuous partial derivatives on Γ′ , and both $\left.\frac{\partial(x,y,z)}{\partial(e,f,g)}\right|_{\Gamma'} \neq 0$ and P,Q,R are continuous on Γ, the equation $\int_L Pdx+Qdy+Rdz = \int_{\Gamma'}(P\frac{\partial x}{\partial e}+Q\frac{\partial x}{\partial e}+R\frac{\partial z}{\partial e})de+(P\frac{\partial x}{\partial f}+Q\frac{\partial y}{\partial f}+R\frac{\partial z}{\partial f})df+(P\frac{\partial x}{\partial g}+Q\frac{\partial y}{\partial g}+R\frac{\partial z}{\partial g})dg$ holds

3.2 Substitution Approach for Surface Integrals

3.2.1 Substitution Technique for Surface Integrals over Area

Theorem 5: Consider a surface segment $\sum$ defined by F(x,y,z) = 0. Through the transformation $\begin{cases}x=x(E,F,G)\\ y=y(E,F,G)\\ z=z(E,F,G)\end{cases}$, the surface $\sum^{'}$ within O – uvw space is mapped bijectively to $\sum$ in O – xyz space. In this scenario, the surface $\sum$possesses first-order continuous partial derivatives x,y,zon $\sum^{'}$, whereas the surface $\sum^{'}$ exhibits first-order continuous partial derivatives ;

$\left.\frac{\partial(x,y,z)}{\partial(E,F,G)}\right|_{\Gamma'} \neq 0$. For the surface integral $\left.(F_x'^2+F_y'^2+F_z'^2)\right|_{\sum} \neq 0$, if the function f is continuous on $\sum$, then $\iint_{\Sigma} f(x,y,z)dS=$

$\iint_{\sum'} f(x,y,z)\left|\frac{\partial(x,y,z)}{\partial(E,F,G)}\right|\sqrt{(F_x'^2+F_y'^2+F_z'^2)}\left\|J\begin{pmatrix}F_x'\\F_y'\\F_z'\end{pmatrix}\right\|^{-1} ds_{EFG}; J=\begin{pmatrix}x_E' & y_E' & z_E'\\ x_F' & y_F' & z_F'\\ x_G' & y_G' & z_G'\end{pmatrix}; ds_{abc}=\sqrt{(dEdF)^2+(dEdG)^2+(dFdG)^2}$

3.2.2 Core Theorem for Coordinate Transformation in Surface Integrals

Theorem 6: Consider $\sum$: T(x,y,z) = 0 as a smooth oriented surface equipped with normal vectors $\{T_x', T_y', T_z')\}$. Under the transformation $\begin{cases}x=x(a,b,c)\\ y=y(a,b,c)\\ z=z(a,b,c)\end{cases}$, examine the oriented surface $\sum'$ within O – uvw.It is assumed that the partial derivatives of the function possess continuity on the domain , and both $\left.\frac{\partial(x,y,z)}{\partial(a,b,c)}\right|_{\Gamma'} \neq 0$ and M,N,K are continuous on $\sum'$, the equation $\iint_{\sum} Mdydz+Ndzdx+Kdxdy=\iint_{\sum'}\begin{pmatrix}M & N & K\\ x_b' & y_b' & z_b'\\ x_c' & y_c' & z_c'\end{pmatrix}dbdc+\begin{pmatrix}M & N & K\\ x_c' & y_c' & z_c'\\ x_a' & y_a' & z_a'\end{pmatrix}dcda+\begin{pmatrix}M & N & K\\ x_a' & y_a' & z_a'\\ x_b' & y_b' & z_b'\end{pmatrix}dadb$

4 Symmetry Method

4.1 Symmetry Properties of Line Integrals

4.1.1 Symmetry Principles for First-Type Line Integrals

Theorem 1: In cases where the curve L possesses symmetrical properties relative to y(x), the relation $\int_L F(x,y)ds=$ $\begin{cases}2*\int_{L_1} K(x,y)ds, K(x,y) \text{ is an even function in term of } x(y)\\ 0, \text{The function } K(x,y) \text{ exhibits odd symmetry with regard to the variable x or y}\end{cases}$ holds where L_1 represents the segment of L defined under $x \geq 0 (y \geq 0)$ specifically the segment of L_1 located in the first quadrant.

Theorem 2: Suppose K(x,y) is continuous on a piecewise smooth curve L. If L possesses central symmetry about the origin, then $\begin{cases}2\int_{L_1} K(x,y)ds\\ 0, else\end{cases}$. Here L_1 denotes the segment of L lying in the right (upper) half-plane.

4.1.2 Symmetry Principles for Second-Type Line

Integrals Theorem 3: Consider L as a piecewise smooth oriented curve in the plane, Aand B are continuous, with L exhibiting symmetry relative to y(x). $\int_L Adx=\begin{cases}2\int_{L_1} Adx.\\ 0, else\end{cases}$

$\int_L Bdx=\begin{cases}2\int_{L_1} Qdy.\\ 0, else\end{cases}$ In this context L_1 represents the segment of L situated in the upper right half-plane.

Proposition 4: Let L denote a piecewise smooth oriented curve lying within the plane , Aand Bare continuous, with L possessing central symmetry about the origin. Then $\int_L A(x,y)dx+B(x,y)dy=$ $\begin{cases}2\int_{L_1} Adx+Bdy\\ 0, else\end{cases}$ In this case L_1 denotes the segment of L located in the upper right half-plane.

4.2 Symmetry Properties of Surface Integrals

Integrals Theorem 5: Consider $\sum$ exhibiting symmetry with respect to xoy(orxoz or*yoz*) plane. Then $\iint_{\Sigma} k(x,y,z)dS=$

$$\begin{cases} 2\iint\limits_{\sum_1} k(x,y,z)\,dS & \sum_1 : \geq 0 (\text{or } y \geq 0 \text{ or } x \geq 0) \\ 0, \text{else} \end{cases}$$

Proposition 6: Consider a piecewise smooth surface $\sum$ possessing symmetry relative to the plane xoy(orxoz or yoz). The segment of $\sum$ lying on the plane xoy(orxoz or yoz) (front or right) is designated as $\sum_1$. Assuming r(x,y,z) is continuous on $\sum$, we have

$$\iint\limits_{\Sigma} r(x,y,z)dxdy = \begin{cases} 2\iint\limits_{\sum_1} r(x,y,z)dxdy \\ 0, \text{else} \end{cases}$$

5 Conclusion

This paper systematically explores three core approaches to handling curve integrals and surface integrals—the definition method, substitution method, and symmetry application—aiming to provide learners with a clear computational framework and optimization strategies. The definition method is suitable for understanding basic concepts, simple integration paths (e. g., straight lines, arcs), and problems with explicit physical meanings (e. g., work done by variable forces, surface mass). Its advantage lies in logical intuition, directly reflecting the essence of integration, and being applicable to all types of curve and surface integrals without additional conditions. However, its drawback is computational complexity, particularly when integration paths or surfaces are complex, where errors in symbols (e. g., second-order integration direction) may lead to deviations. The substitution method is applicable to complex parametric equations or implicit surface problems with obvious symmetry in the integration region or simplifiable through transformations (e. g., polar coordinates, spherical coordinates). Its advantage is significantly simplifying calculations by converting high-dimensional integrals into low-dimensional standard forms, making it suitable for handling irregular regions or abstract integration problems. However, its drawback is requiring high rationality of transformations, such as one-to-one mappings and continuity of partial derivatives, as well as proficiency in advanced knowledge like Jacobi determinants and orthogonal transformations. The symmetry application is suitable for scenarios where the integration region is symmetric (e.g., about coordinate axes, center symmetry) and the integrand has odd or even properties, or when rapid verification and simplified calculations are needed. Its advantage is greatly reducing computational load, even directly yielding zero results, while cultivating geometric intuition and problem simplification skills. However, its drawback is being applicable only to specific symmetry problems, relying on keen judgment of region and function symmetry, with extra attention required for direction and sign matching in second-order integrals. Mastering curve and surface integrals requires balancing conceptual understanding with practical techniques. The definition method forms the foundation, while substitution methods serve as tools, and symmetry offers shortcuts. When learning, we should focus on physical principles to reduce the difficulty of abstract concepts. Through tiered training, we can progressively enhance computational skills and problem-solving abilities, emphasizing the applicable boundaries of methods to avoid mechanical application. This approach not only teaches "how to calculate" but also explains "why it's calculated this way," enabling flexible handling of complex integrals. Note that the three methods discussed here are not isolated—they often require combined use in practice (e. g., first simplifying through symmetry, then calculating with substitution methods). Learners are advised to summarize patterns through practice and develop personalized problem-solving strategies.

References

[1] Tongji University Department of Mathematics. Advanced Mathematics [M]. Higher Education Press, 2007.

[2] Tsinghua University Mathematics Teaching and Research Group. Advanced Mathematics (Volume II) (Basic Part) [M]. Higher Education Press, 2014.

[3] Graduate Mathematics Review Complete Book Compilation Group. Advanced Mathematics Review Guide (Science and Engineering Category) [M]. Southeast University Press, 2003.

[4] Zhao Zhenhua, Zhang Xiaoqiang, Chen Yuzhi. On Training Students' Innovative Thinking Ability in Advanced Mathematics Teaching [J]. Advanced Mathematics Research, 2005.

[5] Zhu Shixin, Tang Shuo, Ning Rongjian, Ren Bei, Zheng Jingbo. Advanced Mathematics (Volume II) [M]. Beijing: Higher Education Press, 2015: 165,175-188,217-233.

[6] Ning Rongjian, Peng Kaijun. Substitution Method for Integral Calculus [J]. University Mathematics, 2016,32(4):123-127.

[7] Song Guozhu, Ren Fuxian, Xu Shaopu, Jiang Dongping. Mathematics Analysis Tutorial (Volume II) [M]. Nanjing: Nanjing University Press, 1992.331-345.